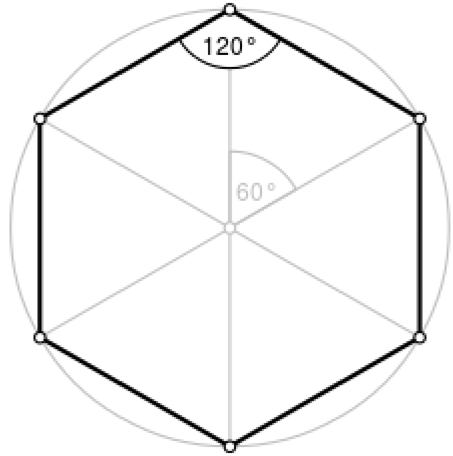
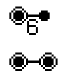
Hexagon

In geometry, a **hexagon** (from Greek ἕξ *hex*, "six" and γωνία, *gonía*, "corner, angle") is a six sided polygon or 6-gon. The total of the internal angles of any hexagon is 720°.

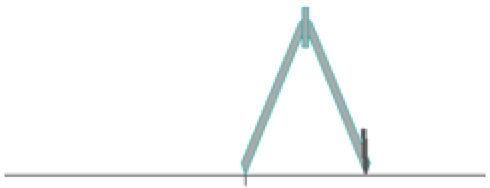
Regular hexagon	
 A regular hexagon	
Type	Regular polygon
Edges and vertices	6
Schläfli symbol	{6}, t{3}
Coxeter diagram	
Symmetry group	Dihedral (D_6), order 2×6
Internal angle (degrees)	120°
Dual polygon	Self

Contents

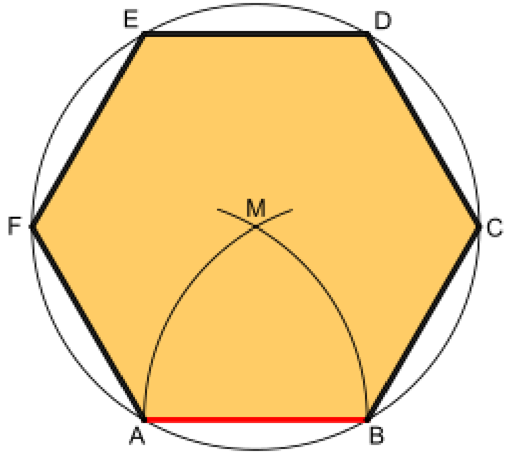
Regular hexagon

Properties	Convex, cyclic, equilateral, isogonal, isotoxal

A *regular hexagon* has Schläfli symbol {6}[1] and can also be constructed as a truncated equilateral triangle, t{3}, which alternates two types of edges.



A step-by-step animation of the construction of a regular hexagon using compass and straightedge, given by Euclid's *Elements*, Book IV, Proposition 15: this is possible as 6 = 2 × 3, a product of a power of two and distinct Fermat primes.



When the side length $\overline{AB}$ is given, then you draw around the point A and around the point B a circular arc. The intersection M is the center of the circumscribed circle. Transfer the line segment $\overline{AB}$ four times on the circumscribed circle and connect the corner points.

A regular hexagon is defined as a hexagon that is both equilateral and equiangular. It is bicentric, meaning that it is both cyclic (has a circumscribed circle) and tangential (has an inscribed circle).

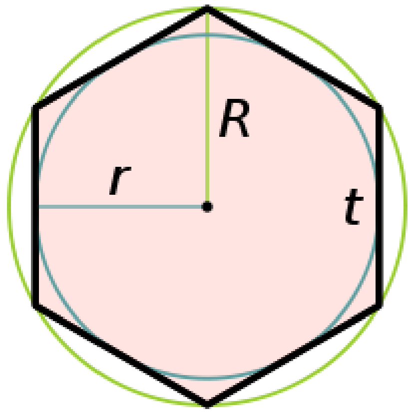
The common length of the sides equals the radius of the circumscribed circle, which equals $\frac{2}{\sqrt{3}}$ times the apothem (radius of the inscribed circle). All internal angles are 120 degrees. A regular hexagon has 6 rotational symmetries (*rotational symmetry of order six*) and 6 reflection symmetries (*six lines of symmetry*), making up the dihedral group D_6. The longest diagonals of a regular hexagon, connecting diametrically opposite vertices, are twice the length of one side. From this it can be seen that a triangle with a vertex at the center of the regular hexagon and sharing one side with the hexagon is equilateral, and that the regular hexagon can be partitioned into six equilateral triangles.

Like squares and equilateral triangles, regular hexagons fit together without any gaps to *tile the plane* (three hexagons meeting at every vertex), and so are useful for constructing tessellations. The cells of a beehive honeycomb are hexagonal for this reason and because the shape makes efficient use of space and building materials. The Voronoi diagram of a regular triangular lattice is the honeycomb tessellation of hexagons. It is not usually considered a triambus, although it is equilateral.

Parameters

The maximal diameter (which corresponds to the long diagonal of the hexagon), *D*, is twice the maximal radius or circumradius, *R*, which equals the side length, *t*. The minimal diameter or the diameter of the inscribed circle (separation of parallel sides, flat-to-flat distance, short diagonal or height when resting on a flat base), *d*, is twice the minimal radius or inradius, *r*. The maxima and minima are related by the same factor:

$$\frac{1}{2}d = r = \cos(30^\circ)R = \frac{\sqrt{3}}{2}R \quad \text{and, similarly, } d = \frac{\sqrt{3}}{2}D$$



The area of a regular hexagon

$$\begin{aligned}A &= \frac{3\sqrt{3}}{2}R^2 = 3Rr = 2\sqrt{3}r^2\\ &= \frac{3\sqrt{3}}{8}D^2 = \frac{3}{4}Dd = \frac{\sqrt{3}}{2}d^2\\ &\approx 2.598R^2 \approx 3.464r^2\\ &\approx 0.6495D^2 \approx 0.866d^2.\end{aligned}$$

For any regular polygon, the area can also be expressed in terms of the apothem, $a = r$, and perimeter, $p = 6R = 4r\sqrt{3}$:

$$\begin{aligned}A &= \frac{ap}{2}\\ &= \frac{r \cdot 4r\sqrt{3}}{2} = 2r^2\sqrt{3}\\ &\approx 3.464r^2,\end{aligned}$$

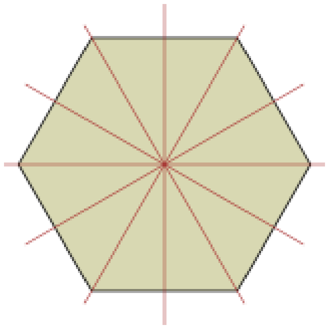
The regular hexagon fills the fraction $\frac{3\sqrt{3}}{2\pi} \approx 0.8270$ of its circumscribed circle.

If a regular hexagon has successive vertices A, B, C, D, E, F and if P is any point on the circumscribing circle between B and C, then PE + PF = PA + PB + PC + PD.

Symmetry

The *regular hexagon* has Dih_6 symmetry, order 12. There are 3 dihedral subgroups: Dih_3, Dih_2, and Dih_1, and 4 cyclic subgroups: Z_6, Z_3, Z_2, and Z_1.

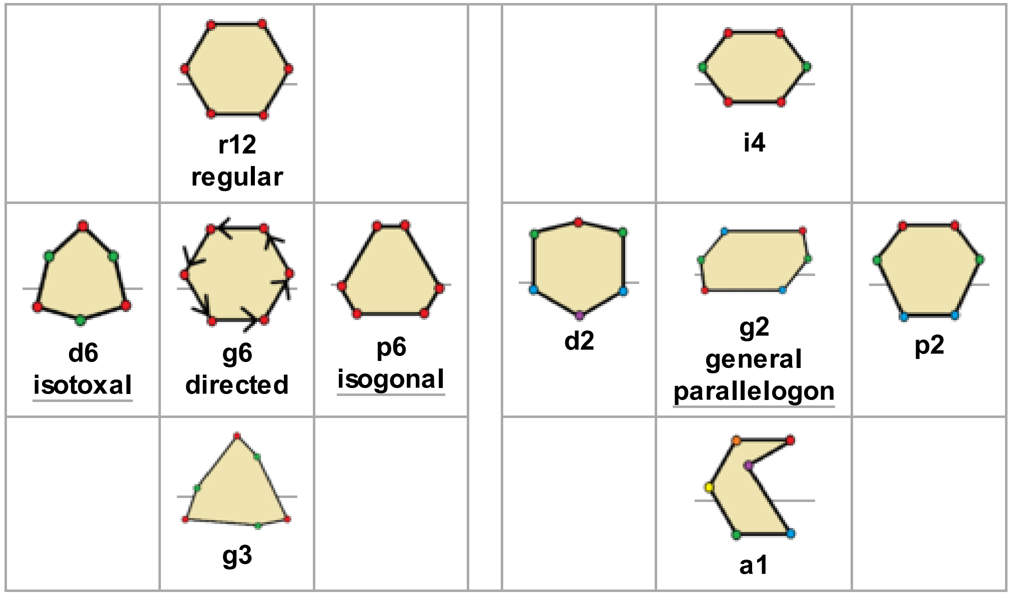
These symmetries express 9 distinct symmetries of a regular hexagon. John Conway labels these by a letter and group order.[2] **r12** is full symmetry, and **a1** is no symmetry. **d6**, a isogonal hexagon constructed by four mirrors can alternate long and short edges, and **p6**, an isotoxal hexagon constructed with equal edge lengths, but vertices alternating two different internal angles. These two forms are duals of each other and have half the symmetry order of the regular hexagon. The **i4** forms are regular hexagons flattened or stretched along one symmetry direction. It can be seen as an elongated rhombus, while **d2** and **p2** can be seen as horizontally and vertically elongated kites. **g2** hexagons, with opposite sides parallel are also called hexagonal parallelogons.

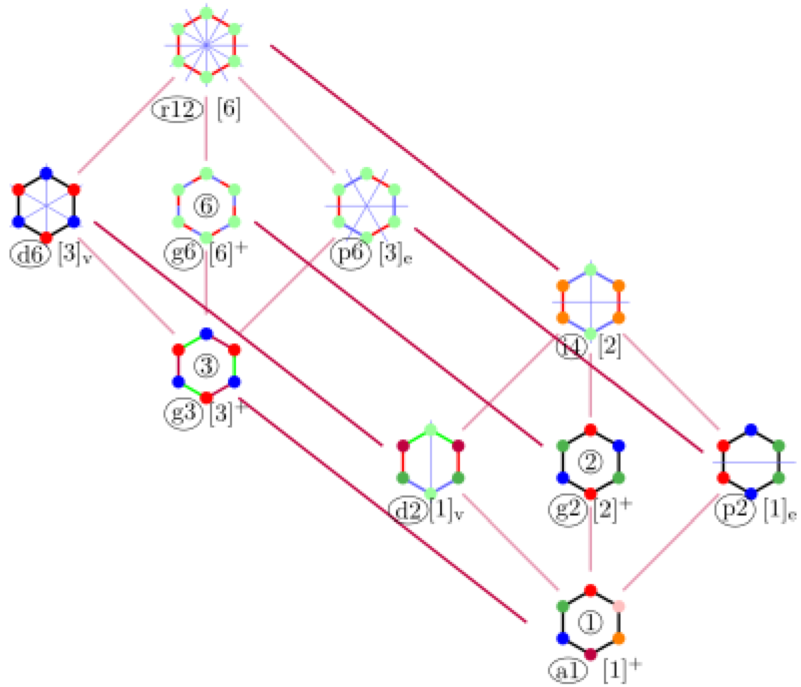


The six lines of reflection of a regular hexagon, with Dih_6 or **r12** symmetry, order 12.

Each subgroup symmetry allows one or more degrees of freedom for irregular forms. Only the **g6** subgroup has no degrees of freedom but can seen as directed edges.

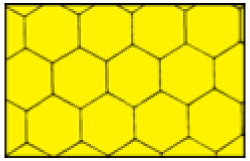
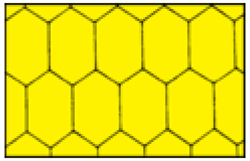
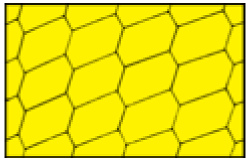
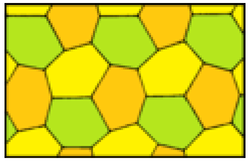
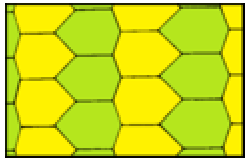
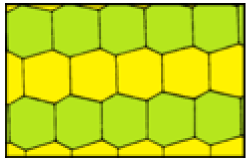
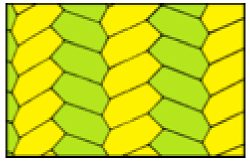
Example hexagons by symmetry





The dihedral symmetries are divided depending on whether they pass through vertices (**d** for diagonal) or edges (**p** for perpendiculars) Cyclic symmetries in the middle column are labeled as **g** for their central gyration orders. Full symmetry of the regular form is **r12** and no symmetry is labeled **a1**.

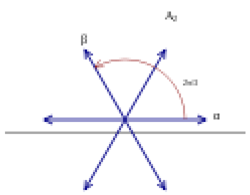
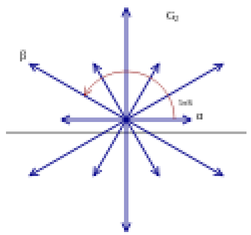
Hexagons of symmetry **g2**, **i4**, and **r12**, as parallelogons can tessellate the Euclidean plane by translation. Other hexagon shapes can tile the plane with different orientations.

p6m (*632)	cmm (2*22)	p2 (2222)	p31m (3*3)	pmg (22*)		pg (××)
						
r12	i4	g2	d2	d2	p2	a1

A2 and G2 groups

The 6 roots of the simple Lie group A2, represented by a Dynkin diagram, are in a regular hexagonal pattern. The two simple roots have a 120° angle between them.

The 12 roots of the Exceptional Lie group G2, represented by a Dynkin diagram are also in a hexagonal pattern. The two simple roots of two lengths have a 150° angle between them.

	
A2 group roots	G2 group roots

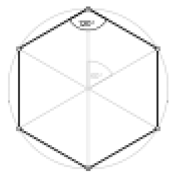
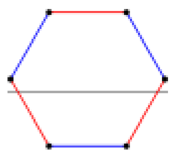
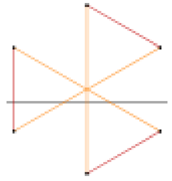
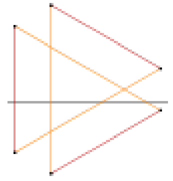
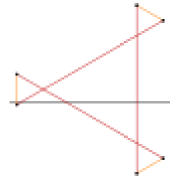
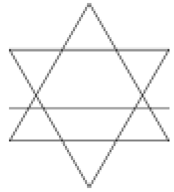
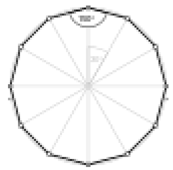
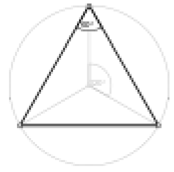
Related polygons and tilings

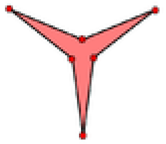
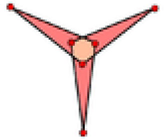
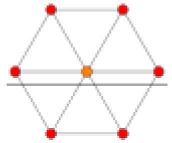
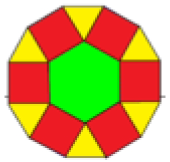
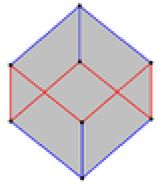
A regular hexagon has Schläfli symbol {6}. A regular hexagon is a part the regular hexagonal tiling, {6,3}, with 3 hexagonal around each vertex.

A regular hexagon can also be created as a truncated equilateral triangle, with Schläfli symbol t{3}. Seen with two types (colors) of edges, this form only has D_3 symmetry.

A truncated hexagon, t{6}, is a dodecagon, {12}, alternating 2 types (colors) of edges. An alternated hexagon, h{6}, is a equilateral triangle, {3}. A regular hexagon can be stellated with equilateral triangles on its edges, creating a hexagram. A regular hexagon can be dissected into 6 equilateral triangles by adding a center point. This pattern repeats within the regular triangular tiling.

A regular hexagon can be extended into a regular dodecagon by adding alternating squares and equilateral triangles around it. This pattern repeats within the rhombitrihexagonal tiling.

							
Regular {6}	**Truncated t{3} = {6}**	**Hypertruncated triangles**			**Stellated Star figure 2{3}**	**Truncated t{6} = {12}**	**Alternated h{6} = {3}**

				
A concave hexagon	**A self-intersecting hexagon (star polygon)**	**Dissected {6}**	**Extended Central {6} in {12}**	**A skew hexagon, within cube**

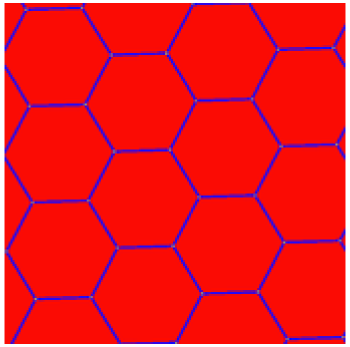
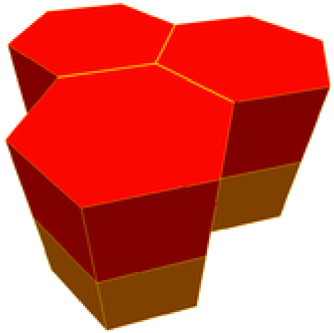
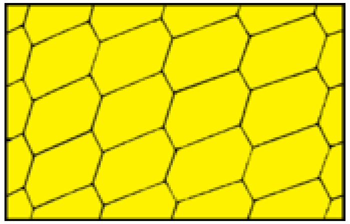
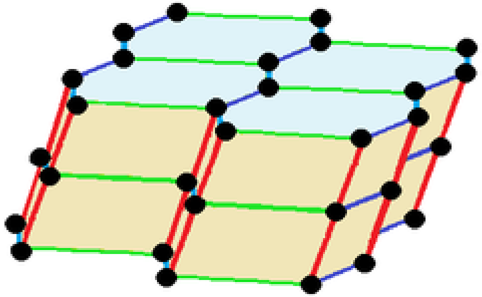
Hexagonal structures

Giant's Causeway closeup

From bees' honeycombs to the Giant's Causeway, hexagonal patterns are prevalent in nature due to their efficiency. In a hexagonal grid each line is as short as it can possibly be if a large area is to be filled with the fewest number of hexagons. This means that honeycombs require less wax to construct and gain lots of strength under compression.

Irregular hexagons with parallel opposite edges are called parallelogons and can also tile the plane by translation. In three dimensions, hexagonal prisms with parallel opposite faces are called parallelohedrons and these can tessellate 3-space by translation.

Hexagonal prism tessellations

Form	Hexagonal tiling	Hexagonal prismatic honeycomb
Regular		
Parallelogonal		

Tesselations by hexagons

In addition to the regular hexagon, which determines a unique tessellation of the plane, any irregular hexagon which satisfies the Conway criterion will tile the plane.

Hexagon inscribed in a conic section

Pascal's theorem (also known as the "Hexagrammum Mysticum Theorem") states that if an arbitrary hexagon is inscribed in any conic section, and pairs of opposite sides are extended until they meet, the three intersection points will lie on a straight line, the "Pascal line" of that configuration.

Cyclic hexagon

The Lemoine hexagon is a cyclic hexagon (one inscribed in a circle) with vertices given by the six intersections of the edges of a triangle and the three lines that are parallel to the edges that pass through its symmedian point.

If the successive sides of a cyclic hexagon are *a*, *b*, *c*, *d*, *e*, *f*, then the three main diagonals intersect in a single point if and only if *ace* = *bdf*.[3]

If, for each side of a cyclic hexagon, the adjacent sides are extended to their intersection, forming a triangle exterior to the given side, then the segments connecting the circumcenters of opposite triangles are concurrent.[4]

If a hexagon has vertices on the circumcircle of an acute triangle at the six points (including three triangle vertices) where the extended altitudes of the triangle meet the circumcircle, then the area of the hexagon is twice the area of the triangle.[5]:p. 179

Hexagon tangential to a conic section

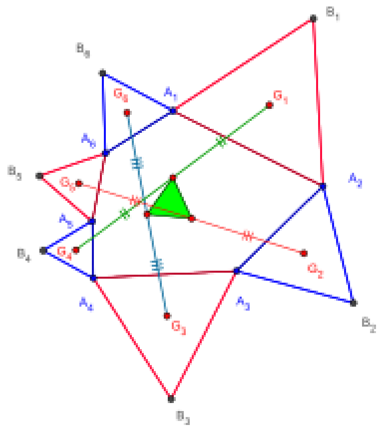
Let ABCDEF be a hexagon formed by six tangent lines of a conic section. Then Brianchon's theorem states that the three main diagonals AD, BE, and CF intersect at a single point.

In a hexagon that is tangential to a circle and that has consecutive sides *a*, *b*, *c*, *d*, *e*, and *f*,[6]

$$a + c + e = b + d + f.$$

Equilateral triangles on the sides of an arbitrary hexagon

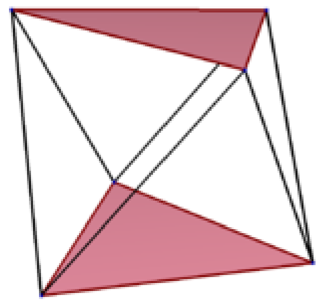
If an equilateral triangle is constructed externally on each side of any hexagon, then the midpoints of the segments connecting the centroids of opposite triangles form another equilateral triangle.[7]:Thm. 1



Equilateral triangles on the sides of an arbitrary hexagon

Skew hexagon

A **skew hexagon** is a skew polygon with 6 vertices and edges but not existing on the same plane. The interior of such an hexagon is not generally defined. A *skew zig-zag hexagon* has vertices alternating between two parallel planes.

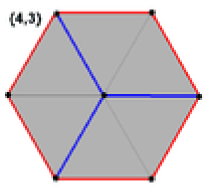
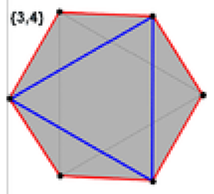


A regular skew hexagon seen as edges (black) of a triangular antiprism, symmetry D_{3d}, $[2^+,6]$, (2*3), order 12.

A **regular skew hexagon** is vertex-transitive with equal edge lengths. In 3-dimensions it will be a zig-zag skew hexagon and can be seen in the vertices and side edges of a triangular antiprism with the same D_{3d}, $[2^+,6]$ symmetry, order 12.

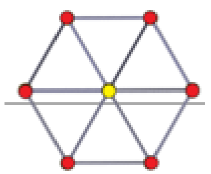
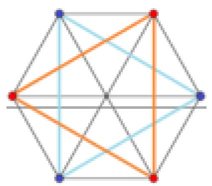
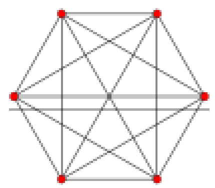
The cube and octahedron (same as triangular antiprism) have regular skew hexagons as petrie polygons.

Skew hexagons on 3-fold axes

 Cube	 Octahedron

Petrie polygons

The regular skew hexagon is the Petrie polygon for these higher dimensional regular, uniform and dual polyhedra and polytopes, shown in these skew orthogonal projections:

4D		5D
 3-3 duoprism	 3-3 duopyramid	 5-simplex

Convex equilateral hexagon

A *principal diagonal* of a hexagon is a diagonal which divides the hexagon into quadrilaterals. In any convex equilateral hexagon (one with all sides equal) with common side a, there exists[8]:p.184,#286.3 a principal diagonal d_1 such that

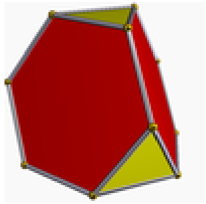
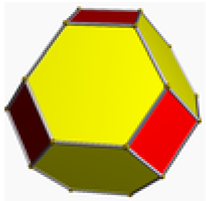
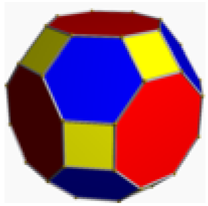
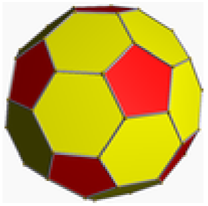
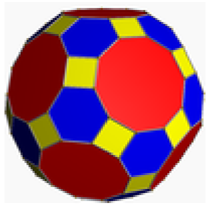
$$\frac{d_1}{a} \leq 2$$

and a principal diagonal d_2 such that

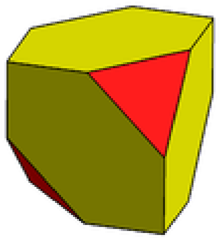
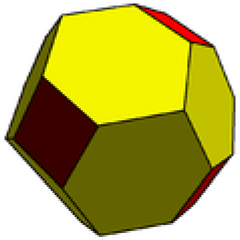
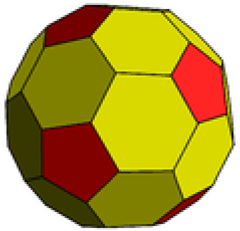
$$\frac{d_2}{a} > \sqrt{3}.$$

Polyhedra with hexagons

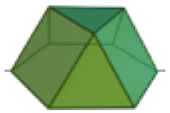
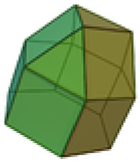
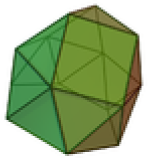
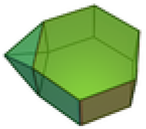
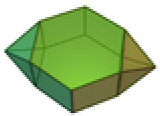
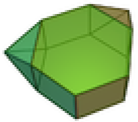
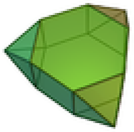
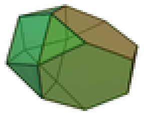
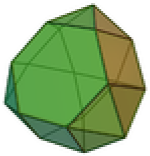
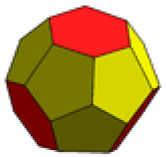
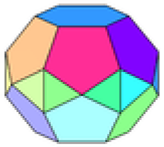
There is no Platonic solid made of only regular hexagons, because the hexagons tessellate, not allowing the result to "fold up". The Archimedean solids with some hexagonal faces are the truncated tetrahedron, truncated octahedron, truncated icosahedron (of soccer ball and fullerene fame), truncated cuboctahedron and the truncated icosidodecahedron. These hexagons can be considered truncated triangles, with Coxeter diagrams of the form and .

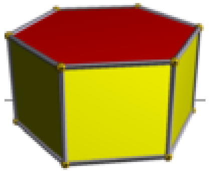
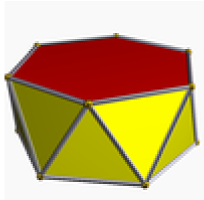
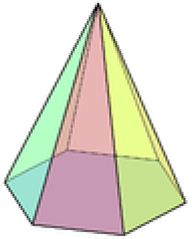
Hexagons in Archimedean solids				
Tetrahedral	**Octahedral**		**Icosahedral**	
 truncated tetrahedron	 truncated octahedron	 truncated cuboctahedron	 truncated icosahedron	 truncated icosidodecahedron

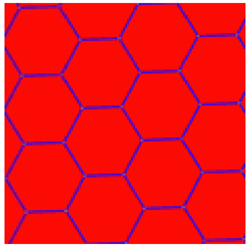
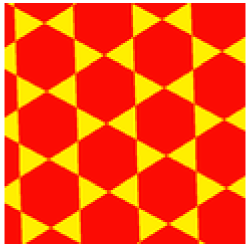
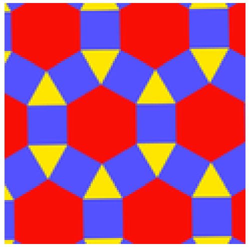
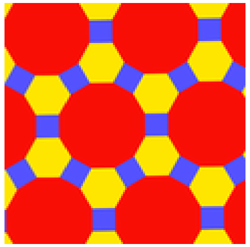
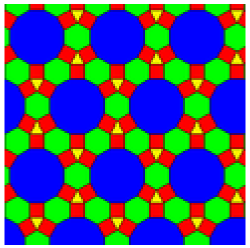
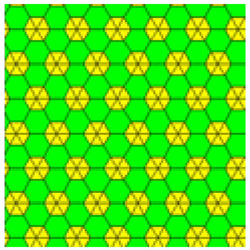
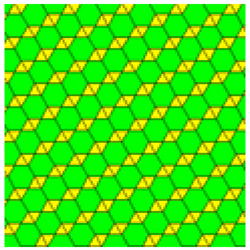
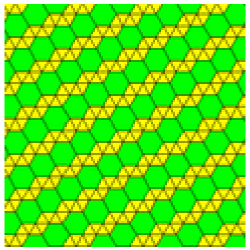
There are other symmetry polyhedra with stretched or flattened hexagons, like these Goldberg polyhedron G(2,0):

Hexagons in Goldberg polyhedra		
Tetrahedral	**Octahedral**	**Icosahedral**
 Chamfered tetrahedron	 Chamfered cube	 Chamfered dodecahedron

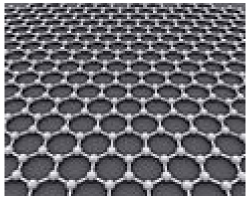
There are also 9 Johnson solids with regular hexagons:

Johnson solids and near-misses with hexagons			
 triangular cupola	 elongated triangular cupola	 gyroelongated triangular cupola	
 augmented hexagonal prism	 parabiaugmented hexagonal prism	 metabiaugmented hexagonal prism	 triaugmented hexagonal prism
 augmented truncated tetrahedron	 triangular hebesphenorotunda	 Truncated triakis tetrahedron	

Prismoids with hexagons		
 Hexagonal prism	 Hexagonal antiprism	 Hexagonal pyramid

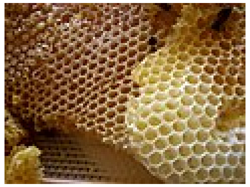
Tilings with regular hexagons			
Regular	1-uniform		
{6,3}	r{6,3}	rr{6,3}	tr{6,3}
			
2-uniform tilings			
			

Hexagons: natural and human-made

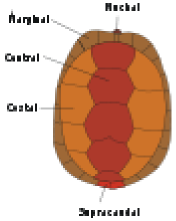


The ideal crystalline structure of graphene is a hexagonal grid.

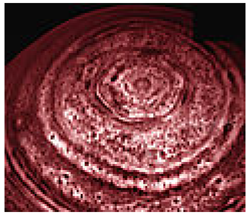
Assembled E-ELT mirror segments



A beehive honeycomb

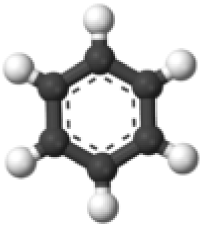


The scutes of a turtle's carapace

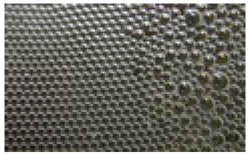


North polar hexagonal cloud feature on Saturn, discovered by Voyager 1 and confirmed in 2006 by Cassini [1] (http://www.nasa.gov/mission_pages/cassini/multimedia/pia09188.html) [2] (http://www.nasa.gov/mission_pages/cassini/media/cassini-20070327.html) [3] (http://adsabs.harvard.edu/cgi-bin/nph-bib_query?bibcode=1988Icar...76..335G&db_key=AST&data_type=HTML&format=)

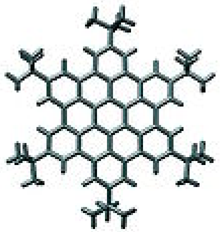
Micrograph of a snowflake



Benzene, the simplest aromatic compound with



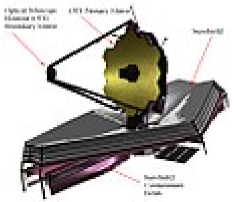
Hexagonal order of bubbles in a foam.



Crystal structure of a molecular hexagon

Naturally formed basalt columns from Giant's

An aerial view of Fort Jefferson in Dry Tortugas



The James Webb Space Telescope mirror is aromatic compound with hexagonal shape.

bubbles in a foam.

molecular hexagon composed of hexagonal aromatic rings reported by Müllen and coworkers in Chem. Eur. J., 2000, 1834-1839.

columns from Giant's Causeway in Northern Ireland; large masses must cool slowly to form a polygonal fracture pattern

Jefferson in Dry Tortugas National Park

Telescope mirror is composed of 18 hexagonal segments.

Metropolitan France has a vaguely hexagonal shape. In French, *l'Hexagone* refers to the European mainland of France aka the "métropole" as opposed to the overseas territories such as Guadeloupe, Martinique or French Guiana.

Hexagonal Hanksite crystal, one of many hexagonal crystal system minerals

Hexagonal barn

The Hexagon, a hexagonal theatre in Reading, Berkshire

Władysław Gliński's hexagonal chess

Pavilion in the Taiwan Botanical Gardens

Hexagonal window

See also

- 24-cell: a four-dimensional figure which, like the hexagon, has orthoplex facets, is self-dual and tessellates Euclidean space

- Hexagonal crystal system
- Hexagonal number
- Hexagonal tiling: a regular tiling of hexagons in a plane
- Hexagram: 6-sided star within a regular hexagon
- Unicursal hexagram: single path, 6-sided star, within a hexagon
- Honeycomb conjecture

References

1. Wenninger, Magnus J. (1974), *Polyhedron Models* (https://books.google.com/books?id=N8lX2T-4njIC&pg=PA9), Cambridge University Press, p. 9, ISBN 9780521098595.
2. John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss, (2008) The Symmetries of Things, ISBN 978-1-56881-220-5 (Chapter 20, Generalized Schaefli symbols, Types of symmetry of a polygon pp. 275-278)
3. Cartensen, Jens, "About hexagons", *Mathematical Spectrum* 33(2) (2000–2001), 37–40.
4. Nikolaos Dergiades, "Dao's theorem on six circumcenters associated with a cyclic hexagon", *Forum Geometricorum* 14, 2014, 243--246. http://forumgeom.fau.edu/FG2014volume14/FG201424index.html
5. Johnson, Roger A., *Advanced Euclidean Geometry*, Dover Publications, 2007 (orig. 1960).
6. Gutierrez, Antonio, "Hexagon, Inscribed Circle, Tangent, Semiperimeter", [4] (http://gogeometry.com/problem/p343_circumscribed_hexagon_tangent_semiperimeter.htm), Accessed 2012-04-17.
7. Dao Thanh Oai (2015), "Equilateral triangles and Kiepert perspectors in complex numbers", *Forum Geometricorum* 15, 105--114. http://forumgeom.fau.edu/FG2015volume15/FG201509index.html
8. *Inequalities proposed in "Crux Mathematicorum"*, [5] (http://www.imomath.com/othercomp/Journ/ineq.pdf).

External links

- Weisstein, Eric W. "Hexagon" (http://mathworld.wolfram.com/Hexagon.html). *MathWorld*.
- Definition and properties of a hexagon (http://www.mathopenref.com/hexagon.html) with interactive animation and construction with compass and straightedge (http://www.mathopenref.com/consthexagon.html).
- Cymatics – Hexagonal shapes occurring within water sound images (http://www.janmeinema.com/cymatics/gallery/gallery_009.html)
- Cassini Images Bizarre Hexagon on Saturn (http://www.nasa.gov/mission_pages/cassini/media/cassini-20070327.html)
- Saturn's Strange Hexagon (http://www.nasa.gov/mission_pages/cassini/multimedia/pia09188.html)
- A hexagonal feature around Saturn's North Pole (http://adsabs.harvard.edu/cgi-bin/nph-bib_query?bibcode=1988Icar...76..335G&db_key=AST&data_type=HTML&format=)
- "Bizarre Hexagon Spotted on Saturn" (http://space.com/scienceastronomy/070327_saturn_hex.html) – from Space.com (27 March 2007)

Fundamental convex regular and uniform polytopes in dimensions 2–10					
Family	A_n	B_n	$I_2(p)$ / D_n	E_6 / E_7 / E_8 / F_4 / G_2	H_n
Regular polygon	Triangle	Square	p-gon	Hexagon	Pentagon
Uniform polyhedron	Tetrahedron	Octahedron • Cube	Demicube		Dodecahedron • Icosahedron
Uniform 4-polytope	5-cell	16-cell • Tesseract	Demitesseract	24-cell	120-cell • 600-cell
Uniform 5-polytope	5-simplex	5-orthoplex • 5-cube	5-demicube		
Uniform 6-polytope	6-simplex	6-orthoplex • 6-cube	6-demicube	1_{22} • 2_{21}	
Uniform 7-polytope	7-simplex	7-orthoplex • 7-cube	7-demicube	1_{32} • 2_{31} • 3_{21}	
Uniform 8-polytope	8-simplex	8-orthoplex • 8-cube	8-demicube	1_{42} • 2_{41} • 4_{21}	
Uniform 9-polytope	9-simplex	9-orthoplex • 9-cube	9-demicube		
Uniform 10-polytope	10-simplex	10-orthoplex • 10-cube	10-demicube		
Uniform *n*-polytope	*n*-simplex	*n*-orthoplex • *n*-cube	*n*-demicube	1_{k2} • 2_{k1} • k_{21}	*n*-pentagonal polytope
Topics: Polytope families • Regular polytope • List of regular polytopes and compounds					

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