Einstein's Paper: "Explanation of the Perihelion Motion of Mercury from General Relativity Theory"

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Abstract

Einstein's original paper "Explanation of the Perihelion Motion of Mercury from General Relativity Theory", 1915, published in German and decades later translated into English, remains hardly accessible for readers. We present the translation recently made by Professor Roger Rydin from the University of Virginia who paid much attention to linguistic fidelity and scientific adequacy of the texts. It is followed with our critical Comments concerning the rigor of Einstein's derivation of the equation of motion and the corresponding approximate solution leading to the perihelion advance formula. The latter was obtained in numerous works later on from the Schwarzschild "exact" solution. Schwarzschild presented it firstly in his letter to Einstein and claimed the formula derived from his solution "identical" to Einstein's one. We draw readers' attention to the fact, however, that some parameters in the Schwarzschild's formula have different physical meanings. This makes formulas, though formally similar, not identical. Yet, one can directly verify that, no matter how the equation is derived, its widely claimed "approximate solution" does not fit the equation.

Key words: Einstein, Schwarzschild, General Relativity, Mercury perihelion, field equations.

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According to general theory of relativity, the elliptical orbit of a planet referred to a Newtonian frame of reference rotates in its own plane in the same direction as the planet moves... The observations cannot be made in a Newtonian frame of reference. They are affected by the precession of the equinoxes, and the determination of the precessional motion is one of the most difficult problems of observational astronomy. It is not surprising that a difference of opinions could exist regarding the closeness of agreement of observed and theoretical motions... I am not aware that relativity is at present regarded by physicists as a theory that may be believed or not, at will. Nevertheless, it may be of some interest to present the most recent evidence on the degree of agreement between the observed and theoretical motions of the planets.

Gerald Clemence (Rev. Mod. Phys. **19**, 361364, 1947)

Erklärung der Perihelbewegung des Merkur aus der allgemeinen Realtivitätstheorie

Von A. Einstein

According to "The Collected Papers of Albert Einstein" (see our *Notes*), this is the lecture given to the Prussian Academy of Sciences in Berlin, 18 November 1915 by A. Einstein. Published 25 November 1915 in *Königlich Preußische Akademie der Wissenschaften (Berlin). Sitzungsberichte (1915): 831-839.*

1 Einstein's paper, 1915

Translation of the paper (along with Schwarzschild's letter to Einstein) by Roger A. Rydin with the following comments by Anatoli A. Vankov

Explanation of the Perihelion Motion of Mercury from General Relativity Theory

Albert Einstein

Introduction

In an earlier version of the work appearing in this journal, I have presented the field equations of gravity, which are covariant under corresponding transformations having a determinant equals unity. In an Addendum to this work, I have shown that each of the field equations is generally covariant when the scalar of the energy tensor of the matter vanishes, and I have thereby shown from the introduction of this hypothesis, through which time and space are robbed of the last vestige of objective reality, that in principle there are no doubts standing against this assertion. [1]

[1]In a soon to follow manuscript, it will be shown that such a hypothesis is unnecessary. It is only important that one such choice of coordinate system is possible, in which the determinant $|g_{\mu\nu}|$ takes the value -1. The following investigation is then independent thereof.

In this work, I found an important confirmation of this radical Relativity theory; it exhibits itself namely in the secular turning of Mercury in the course of its orbital motion, as was discovered by Le Verrier. Namely, the approximately $45''$ per century amount is qualitatively and quantitatively explained without the special hypotheses that he had to assume. [2]

Furthermore, it shows that this theory has a stronger (doubly strong) light bending effect in consequence through the gravitational field than it amounted to in my earlier investigation.

The Gravitational Field

As my last two papers have shown, the gravitational field in a vacuum for a suitably chosen system of coordinates has to satisfy the following

$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} = 0 \tag{1}$$

whereby the quantity $\Gamma^{\alpha}_{\mu\nu}$ is defined through

$$\Gamma^{\alpha}_{\mu\nu} = -\left\{ \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\} = -\sum_{\beta} g^{\alpha\beta} \left\{ \begin{matrix} \mu\nu \\ \beta \end{matrix} \right\} = -\frac{1}{2} \sum_{\beta} g^{\alpha\beta} \left[\frac{\partial g_{\mu\beta}}{\partial x_{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x_{\alpha}} \right] \tag{2}$$

Otherwise, we make the same fundamental hypothesis as in the last paper, that the scalar of the energy tensor of the "material" always vanishes, so that we have the determinant equation

$$|g_{\mu\nu}| = -1 \tag{3}$$

We place a point mass (the Sun) at the origin of the coordinate system. The gravitational field, which this mass point produces, can be calculated from these equations through successive approximations.

In this regard, one may think that the $g_{\mu\nu}$ for the given solar mass is not yet mathematically fully determined through (1) and (3). It follows from it that these equations with the necessary transformation with the determinant equal to unity are covariant. It should be correct in this case to consider

[2] E. Freundlich wrote in an earlier contribution about the impossibility that the anomaly of the motion of Mercury is satisfied on the basis of Newtonian theory, (Astr. Nachr. 4803, Vol. 201, June 1915).

that all these solutions through such transformations can be reduced to one another, that they themselves are also (by given boundary conditions) only formally but not physically distinguishable from one another. These overlying considerations allow me to obtain a solution without considering the question whether or not it is the only unique possibility.

With the above in mind, we go forward. The $g_{\mu\nu}$ is given next in the "*zero*-th approximation" in accord with the Relativity Theory scheme

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Or more compactly

$$g_{\rho\sigma} = -\delta_{\rho\sigma}; \quad g_{\rho 4} = g_{4\rho} = 0; \quad g_{44} = 1 \tag{4a}$$

Hereby, ρ and σ are the indices $1,\ 2,\ 3$: the $\delta_{\rho\sigma}$ is the Kronecker delta symbol equal to 1 or 0, that is when either $\rho = \sigma$ or $\rho \neq \sigma$.

We now set forward the following, that the $g_{\mu\nu}$ differ from the values given in (4a) by an amount that is small compared to unity. This deviation we handle as a small magnitude change of "first order", and functions of n-th degree of this deviation as of "n-th order". Equations (1) and (3) are set in the condition of (4a), for calculation through successive approximations of the gravitational field up to the magnitude n-th order of accuracy. We speak in this sense of the "n-th approximation"; the equations (4a) are the "*zero*-th approximation".

The following given solutions have the following coordinate system-tied properties:

1. All components are independent of x_4.

2. The solution is (spatially) symmetric about the origin of the coordinate system, in the sense that one obtains the same solution if one makes a linear orthogonal (spatial) transformation.

3. The equations $g_{\rho 4} = g_{4\rho} = 0$ are valid exactly (for $\rho = 1,\ 2,\ 3$).

4. The $g_{\mu\nu}$ possess at infinity the values given in (4a).

First Approximation

It is easy to verify, that first order accuracy of the equations (1) and (3) as well as the above named 4 conditions is satisfied through the substitution of

$$g_{\rho\sigma} = -\delta_{\rho\sigma} + \alpha \left(\frac{\partial^2 r}{\partial x_\rho \partial x_\sigma} - \frac{\delta_{\rho\sigma}}{r} \right) = -\delta_{\rho\sigma} - \alpha \frac{x_\rho x_\sigma}{r^3}; \; g_{44} = 1 - \frac{\alpha}{r} \qquad (4b)$$

The $g_{\rho 4}$ as well as $g_{4\rho}$ are thereby set through condition 3; the r means the magnitude of $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

That condition 3 in the sense of first order is fulfilled, one sees at once. In a simple way to visualize that field equation (1) in the first order approximation is also fulfilled, one needs only to observe that the neglect of magnitudes of second and higher orders on the left side of equation (1) can be realized successively through the substitution

$$\sum_\alpha \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha}; \quad \sum_\alpha \frac{\partial}{\partial x_\alpha} \left\{ \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\}$$

whereby α only runs from 1 to 3.

As one sees from (4b), our theory brings with it that in the case of a slowly moving mass the components g_{11} to g_{33} already appear to the non-zero magnitude of first order. We will see later that hereby there is no difference between Newton's law (in the first order approximation). However, it gives a somewhat different influence of the gravitational field on the light ray as in my previous work; as the light velocity is introduced through the equation

$$\sum g_{\mu\nu} dx_\mu dx_\nu = 0 \qquad (5)$$

By use of the Huygens principle, one finds from (5) and (4b) through a simple calculation, that a light ray from the Sun at distance Δ undergoes an angular deflection of magnitude $2\alpha/\Delta$, while the earlier calculation, by which the Hypothesis $\sum T^\mu_\mu = 0$ was not involved, had given the value α/Δ. A corresponding light ray from the surface rim of the Sun should give a deviation of $1.7''$ (instead of $0.85''$). Herein there is no shift of the spectral lines through the gravitational potential, for which Mr. Freundlich has measured the magnitude against the fixed stars, and independently determined that this only depends on g_{44}.

After we have taken $g_{\mu\nu}$ in the first order approximation, we can also calculate the components $\Gamma^{\alpha}_{\mu\nu}$ of the gravitational field in the first order approximation. From (2) and (4b) we obtain

$$\Gamma^{\tau}_{\rho\sigma} = -\alpha \left(\delta_{\rho\sigma} \frac{x_\tau}{r^3} - \frac{3x_\rho x_\sigma x_\tau}{2r^5} \right) \qquad (6a)$$

where ρ, σ, τ take on the values 1, 2, 3, and

$$\Gamma^{\sigma}_{44} = \Gamma^{4}_{4\sigma} = -\frac{\alpha x_\sigma}{2r^3} \qquad (6b)$$

whereby σ takes on the values 1, 2, 3. The single components in which the index 4 appears once or three times, vanish.

Second Approximation

It will be shown here that we only need the three components Γ^{σ}_{44} accurate in the magnitude of second order to be able to evaluate the planetary orbit with sufficient accuracy. For this, it is enough to use the last field equation together with the general conditions, which have led to our general solution. The last field equation

$$\sum_{\sigma} \frac{\partial \Gamma^{\sigma}_{44}}{\partial x_\sigma} + \sum_{\sigma\tau} \Gamma^{\sigma}_{4\tau} \Gamma^{\tau}_{4\sigma} = 0$$

goes with reconsideration of (6b) by neglect of magnitudes of third and higher orders over to

$$\sum_{\sigma} \frac{\partial \Gamma^{\sigma}_{44}}{\partial x_\sigma} = -\frac{\alpha^2}{2r^4}$$

From here follows, with reconsideration of (6b) and the symmetry properties our solution

$$\Gamma^{\sigma}_{44} = -\frac{\alpha x_\sigma}{2r^3} \left(1 - \frac{\alpha}{r} \right) \qquad (6c)$$

Planetary Motion

From the General Relativity theory motion equations of a material point in a strong field, we obtain

$$\frac{d^2 x_\nu}{ds^2} = \sum_{\sigma\tau} \Gamma^{\nu}_{\sigma\tau} \frac{dx_\sigma}{ds} \frac{dx_\tau}{ds} \qquad (7)$$

From this equation, it follows that the Newton motion equation is obtained as a first approximation. Namely, when the speed of a point particle is small with respect to the speed of light, so dx_1 , dx_2 , dx_3 are small against dx_4. It follows that we come to the first approximation, in which we take on the right side always the condition $\sigma = \tau = 4$. One obtains then with consideration of (6b)

$$\frac{d^2x_\nu}{ds^2} = \Gamma^\nu_{44} = -\frac{\alpha x_4}{2r^3}; \quad (\nu = 1,\ 2,\ 3); \quad \frac{d^2x_4}{ds^2} = 0 \qquad (7a)$$

These equations show that one can take as a first approximation $s = x_4$. Then the first three equations are accurately Newtonian. This leads one to the planar orbit equations in polar coordinates r, ϕ, and so leads to the known energy and the Law of Area equations

$$\frac{1}{2}u^2 + \Phi = A; \quad r^2\frac{d\phi}{ds} = B \qquad (8)$$

where A and B are constants of the energy- as well as Law of Areas, whereby the shortened form is inserted.

$$\Phi = -\frac{\alpha}{2r}; \quad u^2 = \frac{dr^2 + r^2 d\phi^2}{ds^2} \qquad (8a)$$

We now have the Equations (7) evaluated to an accurate magnitude. The last of Equations (7) then leads together with (6b) to

$$\frac{d^2x_4}{ds^2} = 2\sum_\sigma \Gamma^4_{\sigma 4}\frac{dx_\sigma}{ds}\frac{dx_4}{ds} = -\frac{dg_{44}}{ds}\frac{dx_4}{ds}$$

or in magnitude of first order exactly to

$$\frac{dx_4}{ds} = 1 + \frac{\alpha}{r} \qquad (9)$$

We now go to the first of the three equations (7). The right side becomes

a) for the index combination $\sigma = \tau = 4$

$$\Gamma^\nu_{44}\left(\frac{dx_4}{ds})\right)^2$$

or with reconsideration of (6c) and (9) in magnitude of second order exactly

$$-\frac{\alpha x_\nu}{2r^3}\left(1+\frac{\alpha}{r}\right)$$

b) with reconsideration thereof for the index combination $\sigma \neq 4$, $\tau \neq 4$ (which alone still comes into consideration), and the fact that the products

$$\frac{dx_\nu}{ds}\frac{dx_\tau}{ds}$$

with reconsideration of (8) are seen as magnitudes of first order, [3] and are likewise accurate to second order, we obtain

$$-\frac{\alpha x_\nu}{r^3}\sum_{\sigma\tau}\left(\delta_{\sigma\tau}-\frac{3x_\sigma x_\tau}{2r^2}\right)\frac{dx_\sigma}{ds}\frac{dx_\tau}{ds}$$

The summation gives

$$-\frac{\alpha x_\nu}{r^3}\left[u^2-\frac{3}{2}\left(\frac{dr}{ds}\right)^2\right]$$

In hindsight therefore, one obtains for the equations of motion in magnitude of second order the exact form

$$\frac{d^2x_\nu}{ds^2}=-\frac{\alpha x_\nu}{2r^3}\left[1+\frac{\alpha}{r}+2u^2-3\left(\frac{dr}{ds}\right)^2\right] \qquad (7b)$$

which together with (9) determines the motion of the point mass. Besides, it should be remarked that (7b) and (9) for the case of an orbital motion give no deviation from Kepler's third law.

From (7b) next follows the exactly valid form of the equation

$$r^2\frac{d\phi}{ds}=B \qquad (10)$$

where B means a constant. The Law of Areas is also accurate in the magnitude of second order, when one uses the "period" of the planet for the

[3]This result we can interpret from the field components $\Gamma^\nu_{\sigma\tau}$ with insertion in equation (6a) of the first order approximation.

time measurement. To now obtain the secular advance of the orbital ellipse from (7b), one inserts the members of first order in the brackets of the right side arranging it to best advantage using (10), and in the first term of the equations (8), through which operation the members of second order on the right side are not changed. Through this, the brackets take the form

$$\left[1 - 2A + \frac{3B^2}{r^2}\right]$$

Finally, one chooses $s\sqrt{1-2A}$ as the second variable, and again calls it s, so that one has a slightly changed meaning of the constant B:

$$\frac{d^2x_\nu}{ds^2} = -\frac{\partial\Phi}{dx_\nu}; \quad \Phi = -\frac{\alpha}{2r}\left[1 + \frac{B^2}{r^2}\right] \tag{7c}$$

By the determination of the orbital form, one now goes forth exactly as in the Newtonian case. From (7c) one next obtains

$$\frac{dr^2 + r^2 d\phi^2}{ds^2} = 2A - 2\Phi$$

One eliminates ds from this equation with the help of (10), and so obtains, in which one designates by x the magnitude $1/r$:

$$\left(\frac{dx}{d\phi}\right)^2 = \frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3 \tag{11}$$

which equation distinguishes itself from the corresponding Newtonian theory only through the last member on the right side.

That contribution from the radius vector and described angle between the perihelion and the aphelion is obtained from the elliptical integral

$$\phi = \int_{\alpha 1}^{\alpha 2} \frac{dx}{\sqrt{\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3}}$$

where α_1 and α_2 are the corresponding first roots of the equation

$$\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3 = 0$$

which means, the very close neighboring roots of the equation corresponding to leaving out the last term member.

Hereby we can with reasonable accuracy replace it with

$$\phi = \left[1 + \frac{\alpha}{2}(\alpha_1 + \alpha_2)\right] \int_{\alpha 1}^{\alpha 2} \frac{dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)(1-\alpha x)}}$$

or after expanding of $(1 - \alpha x)^{(-1/2)}$:

$$\phi = \left[1 + \frac{\alpha}{2}(\alpha_1 + \alpha_2)\right] \int_{\alpha 1}^{\alpha 2} \frac{(1 + \frac{\alpha}{2}x)dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}}$$

The integration leads to

$$\phi = \pi \left[1 + \frac{3}{4}\alpha(\alpha_1 + \alpha_2)\right]$$

or, if one takes α_1 and α_2 as reciprocal values of the maximal and minimal distance from the Sun,

$$\phi = \pi \left[1 + \frac{3\alpha}{2a(1-e^2)}\right] \tag{12}$$

For an entire passage, the perihelion moves by

$$\epsilon = 3\pi \left[\frac{\alpha}{a(1-e^2)}\right] \tag{13}$$

in the directional sense of the orbital motion, when we designate by a the major half axis, and by e the eccentricity. This leads one to the period T (in seconds), so one obtains with c as light velocity in cm/sec:

$$\epsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1-e^2)} \tag{14}$$

This calculation leads to the planet Mercury to move its perihelion forward by 43 $''$ per century, while the astronomers give 45 $''\pm 5''$, an exceptional difference between observation and Newtonian theory. This has great significance as full agreement.

For Earth and Mars the astronomers give a forward movement of 11″ and 9″ respectively per century, while our formula gives only 4″ and 1″, respectively. It appears however from these results, considering the small eccentricity of the orbits of each planet, a smaller effect is appropriate. Confirmation for the correctness of these values for the movement of the perihelion is the product with the eccentricity $(e\frac{d\pi}{dt})$.

	$(e\frac{d\pi}{dt})''$
Mercury	8.48 ± 0.43
Venus	-0.05 ± 0.25
Earth	0.10 ± 0.13
Mars	0.75 ± 0.35

One considers for these the magnitudes of the Newcomb given values, which I thank Dr. Freundlich for supplying, so one gains the impression that only the forward movement of the perihelion of Mercury will ever be truly proven. I will however gladly allow professional astronomers a final say.

End of the paper

2 Schwarzschild's letter to Einstein

Letter from K Schwarzschild to A Einstein dated 22 December 1915

The letter is presented in English owing to Professor Roger A. Rydin

Honored Mr. Einstein,

In order to be able to verify your gravitational theory, I have brought myself nearer to your work on the perihelion of Mercury, and occupied myself with the problem solved with the First Approximation. Thereby, I found myself in a state of great confusion. I found for the first approximation of the coefficient $g_{\mu\nu}$ *other than your solution the following two:*

$$g_{\rho\sigma} = -\frac{\beta x_\rho x_\sigma}{r^5} + \delta_{\rho\sigma}\left(\frac{\beta}{3r^3}\right); \quad g_{44} = 1$$

As follows, it had beside your α *yet a second term, and the problem was physically undetermined. From this I made at once by good luck a search for a full solution. A not too difficult calculation gave the following result: It gave only a line element, which fulfills your conditions 1) to 4), as well as field- and determinant equations, and at the null point and only in the null point is singular.*

If:

$$x_1 = r\cos\phi\cos\theta, \quad x_2 = r\sin\phi\cos\theta, \quad x_3 = r\sin\theta$$

$$R = (r^3+\alpha^3)^{1/3} = r\left(1 + \frac{1}{3}\frac{\alpha^3}{r^3} + ...\right)$$

then the line element becomes:

$$ds^2 = \left(1 - \frac{\gamma}{R}\right)dt^2 - \frac{dR^2}{\left(1-\frac{\gamma}{R}\right)} - R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

R, θ, ϕ *are not "allowed" coordinates, with which one must build the field equations, because they do not have the determinant = 1, however the line element expresses itself as the best.*

The equation of the orbit remains exactly as you obtained in the first approximation (11), only one must understand for x *not* $1/r$*, but* $1/R$*, which is a difference of the order of* 10^{-12}*, so it has practically the same absolute validity.*

The difficulty with the two arbitrary constants α and β*, which the First Approximation gave, resolves itself thereby, that* β *must have a determined value of the order of* α^4 *, so as* α *is given, so will the solution be divergent by continuation of the approximation.*

It is after all the clear meaning of your problem in the best order.

It is an entirely wonderful thing, that from one so abstract an idea comes out such a conclusive clarification of the Mercury anomaly.

As you see, it means that the friendly war with me, in which in spite of your considerable protective fire throughout the terrestrial distance, allows this stroll in your fantasy land.

2.1 Comments

2.1.1 Historical remarks

Einstein's paper devoted to the GR prediction of Mercury's perihelion advance, *Doc.*24 (see *Notes*), is the only one among his publications that contains the explanation of the GR effect. In his following paper *The Foundations of the General Theory of Relativity*, 1916, *Doc.*30, Einstein presents his new (he called it "correct") calculation of the bending of light while the Mercury perihelion is only mentioned by referring it as in *Doc.*24, along with Schwarzshild's work on "the exact solution". Since then, to our knowledge, he never returned to the methodology of the GR perihelion advance problem.

Meanwhile, numerous works have been published and continue to appear in press with suggestions of "clarification", "improvement", "radical change" or "refutation" of Einstein's 1915 prediction of the perihelion advance and bending of light. Most of them, in our view, are results of either confusion or lack of qualification. Qualified works presenting a fresh view of the problem, or related new ideas or concepts are discussed in [Van10] (see *Notes*). Among them, there is a monograph on General Relativity by Bergmann (1942) with Foreword by A. Einstein who clearly acknowledged his own advisory and authorization role in the book composition. Strangely enough, the fact that given there derivations of the GR predictions are quite different from those in *Doc.*24, is not paid much attention in the literature. In spite of methodological differences, claimed predictions in the above works, however, remain the same.

As a matter of fact, the GR foundational premises have been subjected to changes and reinterpretations (optional, alternative, or claimed "correct" ones) by Einstein himself, his advocates as well as today's GR specialists and self-proclaimed "experts". Among the key issues, the problems of energy and angular momentum conservation along with the properties of stress-energy-momentum tensor remain the "hot" (better say, controversial) ones. Possibly, this is one of the reasons why there are numerous publications devoted to the GR perihelion advance effect and the light bending, which are considered controversial or arguable.

2.1.2 Einstein and Schwarzschild "in a friendly war"

One may be critical of Einstein's work (*Doc.*24) in many respects, some arguable issues are worth noting here.

1. The difference from Newton's physics is an appearance of the GR term in the equation of motion (11), not to speak about differentials with respect to the proper time τ. In operational terms, the proper time is recorded by a clock attached to the test particle moving along the world line s so that components $dx^\mu/ds = u^\mu$ define a tangential 4-velocity unit vector. In SR, it relates to the proper 4-momentum vector $P^\mu = mu^\mu$. In *Doc.*24, however, the proper time is actually interpreted similarly to the coordinate ("far-away") time $t = \gamma\tau$, which defines three components of velocity $v_i = dx_i/dt$ ($i = 1,\ 2,\ 3$). Thus, the Lorentz factor in the relativistic kinetic energy becomes lost.

2. The equation of motion (11) (the objective of the work) is obtained at the expense of an arbitrary replacement of $s = c\tau$ with $s\sqrt{1-2A}$ where a difference of the factor $\sqrt{1-2A}$ from unity has a magnitude of the order α/r_0. The corresponding impact on the solution is of the order of the effect in question. At the same time, the GR conservation laws for total energy and angular momentum become controversial: with τ replaced with t, both laws formally appear in the Newtonian form.

3. Unlike in the paper (*Doc.*24), in Bergman's book the GR term becomes responsible for both the perihelion advance and the bending of light; consequently, the derivations of both effects principally changed, first of all, the Schwarzschild metric was acknowledged as the theoretical basis for the GR effects evaluation. It should be noted that, while the methodology changed, the angular momentum ("area") law remained unaffected by the GR term. One can argue, however, that the GR perihelion advance effect necessarily requires a relativistic generalization of the classical conservation laws.

The immediate response to Einstein's work (*Doc.*24) came from Karl Schwarzschild. His famous work "On the Gravitational Field of a Point-Mass, According to Einstein's Theory", *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften, 189-196, 1916)* was published less than a month after Einstein's work. It should be noted that Schwarzschild derived "the exact solution" in the form to be consistent with the Einstein's "four conditions". It has no central divergence and cannot be mixed up with the commonly known "Schwarzschild metric". The less known fact is that Schwarzschild, before the publication, wrote a letter to Einstein in which he criticized Einstein for mistakes in "the successive approximation approach", as is seen from the above translated *Letter from K Schwarzschild to A. Einstein* dated 22 December 1915, in *ColPap*, vol. 8a, *Doc*.169.

Schwarzschild claims that his solution reproduces Einstein's prediction of the GR perihelion advance. But it does not, – because the concept of energy is gone, and coefficients of the equation have very different meaning there. Let us see it in more details.

Schwarzschild's original "exact unique solution" to Einstein's field equations in vacuum for a point source is the squared 4-world line element, which in the polar coordinates is given by

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 d\phi^2 \tag{2.1}$$

where

$$R = (r^3 + \alpha^3)^{1/3}$$

The α is the so-called Schwarzschild's radius equal to the doubled gravitational radius r_g, that is, $\alpha = 2r_g = 2GM/c_0^2$;

the r is the coordinate radial distance, not strictly determined but used as a measure of distance from the source.

the t is the coordinate ("far-away") time.

the R is the approximation of r to the precision of α^3 in the original Schwarzschild's metric.

At some historical moment, the Schwarzschild metric in literature acquired the form, in which the R was replaced with the r, so that the metric became divergent. The original metric, as mentioned, does not diverge: $R \to \alpha$ as $r \to 0$. This means that, strictly speaking, "the black hole" concept does not come out from the General Relativity theory (see Stephen J. Crothers on the Internet and elsewhere). As is known, Einstein himself resisted the BH idea. He also understood, when speculating about the gravitational radiation (particularly, from a BH neighborhood), that a linear perturbative approximation to the metric does not provide it. As concerns the perihelion advance, Einstein seemed to be quite reluctant to appreciate "the exact solution" and the way how Schwarzschild treated it, in spite of the fact that the Schwarzschild metric respects all four Einstein's conditions.

Schwarzschild traditionally starts with the three standard integrals of particle motion:

$$\left(1-\frac{\alpha}{R}\right)\left(\frac{dt^2}{ds^2}\right)-\left(1-\frac{\alpha}{R}\right)^{-1}\left(\frac{dR^2}{ds^2}\right)-R^2\left(\frac{d\phi^2}{ds^2}\right)=const=h \quad (2.2)$$

$$R^2\left(\frac{d\phi}{ds}\right)=c \quad (2.3)$$

$$\left(1-\frac{\alpha}{R}\right)\left(\frac{dt}{ds}\right)=const=1 \quad (2.4)$$

The integral (2.3) coincides with Einstein's expression for the angular momentum. However, (2.4) is not the energy: according to Schwarzschild, it is "the definition of the unit of time". This is in the spirit of the General Relativity Theory, in which the energy is not locally defined. Consequently, the constant (2.2) denoted h must be $h = 1$ by construction.

After that, the equation of motion for $x = 1/R$ takes the form

$$\left(\frac{dx}{d\phi}\right)^2=\frac{1-h}{c^2}+\frac{h\alpha}{c^2}-x^2+\alpha x^3 \quad (2.5)$$

where the first term on the right sight is zero (compare it with Einstein's equation (11)).

3 Critique

Now, we are going to follow Einstein's solution step-by-step. Unfortunately, Einstein wrote the paper (*Doc.*25) very schematic, so that one needs to restore his line of thoughts and, at the same time, correct mistakes (see *Notes*).

The exact solution to (11) is given by:

$$\phi=\int_{x1}^{x2}\frac{dx}{\sqrt{\alpha(x-x_1)(x-x_2)(x-x_3)}} \quad (3.1)$$

Here we denote $x_1 = 1/r_1$, $x_2 = 1/r_2$, $x_3 = 1/r_3$ (instead of denotations $\alpha_1 = 1/r_1$, $\alpha_2 = 1/r_2$, $\alpha_3 = 1/r_3$ in *Doc.*24), which are real roots of the homogeneous cubic equation of the bounded motion problem (11). Let the third root x_3 be due to the GR term. The integral cannot be calculated analytically (though it can be expressed in terms of elliptic functions). Einstein's obvious idea is to compare (3.1) with the analogous solution in the classical (Newtonian) formulation of the problem, the GR "small" term being considered a perturbation source.

Einstein's approach to (3.1) is to eliminate x_3 using an exact relationship between the three roots:

$$(x_1 + x_2 + x_3) = \frac{1}{\alpha} \tag{3.2}$$

Recall, the $\alpha = 2r_g$ where r_g is the gravitational radius. He begins with the approximation to (3.2) by putting $x_3 = 1/\alpha$ with the following algebraic approximation to (3.1) (to the precision of order α^2) allowing one to split the integrand into two additive parts: $I(x) = I_1(x) + I_2(x)$, where I_1 is the main part from the classical solution, and I_2 is the part due to a perturbation of the classical solution by the GR term:

$$\frac{1}{\sqrt{-\alpha(x - x_1)(x - x_2)(x_3 - x)}} \approx$$

$$\frac{1}{\sqrt{-(x - x_1)(x - x_2)}} + \frac{\alpha(x_1 + x_2 + x)}{2\sqrt{-(x - x_1)(x - x_2)}} \tag{3.3}$$

As wished, (3.3) does not have the root x_3 and is easily integrated analytically.

The final crucial step to the GR solution must be an evaluation of the impact of the GR term on the roots from the GR solution in comparison with the corresponding classical roots $\tilde{x}_1$, $\tilde{x}_2$ describing a "nearly circular" orbit when $\tilde{x}_1 + \tilde{x}_2 = 2/\tilde{r}_0$. The radius $\tilde{r}_0$ is the one of the classical circle to be compared with the corresponding r_0 from the GR exact solution. In the *zero*-th approximation, it is assumed that the impact of the GR term on the Newtonian solution is negligible.

Considering the effective potential in both cases, it is easy to find the relationship $[Van10]$

$$r_0 = \left(\tilde{r}_0 - \frac{3}{2}\alpha\right) \tag{3.4}$$

Here, $r_0 >> r_3$, or equivalently $x_0 = 1/r_0 << x_3$, and $r_3 = \alpha(1 + 3\alpha/2)$. It should be immediately noticed that the above radial difference in a circular and nearly circular motion translates into the corresponding difference in circumferences, $3\pi\alpha$, that is exactly the GR perihelion advance effect. A fatal mistake in final Einstein's solution in the problem of planetary perihelion advance is the assumption that the impact of the GR term on the roots x_1 and x_2 is negligible. Let us deliberate the problem in more details.

Let us assume for a moment that $r_1 = \tilde{r}_1$, $r_2 = \tilde{r}_2$, consequently, $r_0 = \tilde{r}_0$. Then the standard integration of I_1 over half a period (from r_1 to r_2) makes π. The similar analytical integration of I_2 gives

$$\Delta\theta = \frac{3\pi\alpha}{4}\pi\alpha(x_1 + x_2) \tag{3.5}$$

The final result would be the angular (perihelion) advance per one revolution given by

$$\Delta\theta = \frac{3}{2}\pi\alpha(x_1 + x_2) \tag{3.6}$$

or, for a circular orbit, $\Delta\theta = 3\pi\alpha/r_0 = 6\pi r_g/r_0$. There $(x_1 + x_2) \approx (2/r_0)$ while the eccentricity $e << 1$. The approximate analytical solution to the equation (11) is supposed to be obtained for initial conditions analogous to that in the corresponding classical problem and as a result of a small perturbation of physical parameters such as potential, kinetic and total energy as well as the angular momentum. At the same time, a mathematical approximation is made due to the assumption that the impact of the GR term on the classical roots x_1 and x_2 is negligible. As a result, the solution could be understood as a periodic, not closed, orbit with a period $\nu\theta = 2\pi$ (see *Bergmann*): $x(\theta) = (1/r_0)(1 + e\cos\nu\theta)$, or

$$r(\theta) = \frac{r_0}{(1 + e\cos\nu\theta)} \tag{3.7}$$

where $\nu \approx (1 - 3r_g/r_0)$ is a factor of the deficit of full angular rotation in one classical revolution. To complete the revolution, a planet needs to rotate through the additional angle $2\pi(3r_g/r_0)$, which should be observed, in Clemence's terms, in the inertial frame as a non-classical effect of rotation of orbital plane in the direction of planet motion (the claimed GR perihelion advance effect).

The fact of a physical inconsistence of the prediction (3.7) can be verified by a substitution of the solution into the original Einstein's equation (11). The latter in the parametric form with geometrical parameters such as eccenticity e and semilatus rectum p is given by

$$\left(\frac{1}{\nu^2}\right)\left(\frac{dx}{d\theta}\right)^2 = -\frac{(1-e^2)}{p^2} + \frac{2x}{p} - x^2 + \boxed{2r_g x^3} \qquad (3.8)$$

where the GR term is framed.

One immediately finds that the solution (3.7) satisfies the equation (3.8) if and only if the GR term is removed from the equation. Once it is removed, *any* value of ν does perfectly fit the equation.

It is obvious that the effect $\Delta\theta = 2\pi(3r_g/r_0)$ is (falsely) originated because of the radial shift $\tilde{r}_0 - r_0 = 3r_g$ not taken into account. Let us consider an almost circular orbit of a radius $\tilde{r}_0$ in the classical case and the radius r_0 in the GR case under similar conditions. The fact is that the GR term makes a circumference of the circular orbit shorter by $3\pi\alpha$, which in turn makes a deficit of angle of rotation $\Delta\pi = 6\pi r_g/r_0$. Therefore, in (3.3) we have to drop the wrong assumption of the equality $\tilde{r}_0 = r_0$ and account for the actual shortage of circumference in the first (presumably, "classical") integral. It makes the angle less than the expected value of π in a half a period, namely:

$$\frac{1}{2}\Delta\theta_1 = \int_{\tilde{r}_1}^{\tilde{r}_2 - \Delta r} I_1(r)dr = \pi - \frac{3}{2}\pi\alpha \qquad (3.9)$$

The second (perturbation) integral is not sensitive to the radius alternation and gives a result (in agreement with Einstein's result)

$$\frac{1}{2}\Delta\theta_2 = 3\pi\alpha \qquad (3.10)$$

which makes a total π, that is, a zero angular advance.

A Einstein's original paper, 1915

Erklärung der Perihelbewegung des Merkur aus der allgemeinen Realtivitätstheorie

Von A. Einstein

Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie.

Von A. Einstein.

In einer jüngst in diesen Berichten erschienenen Arbeit, habe ich Feldgleichungen der Gravitation aufgestellt, welche bezüglich beliebiger
[1] Transformationen von der Determinante 1 kovariant sind. In einem Nachtrage habe ich gezeigt, daß jenen Feldgleichungen allgemein kovariante entsprechen, wenn der Skalar des Energietensors der »Materie« verschwindet, und ich habe dargetan, daß der Einführung dieser Hypothese, durch welche Zeit und Raum der letzten Spur objektiver Realität beraubt werden, keine prinzipiellen Bedenken ent-
[2] gegenstehen[1].

In der vorliegenden Arbeit finde ich eine wichtige Bestätigung dieser radikalsten Relativitätstheorie; es zeigt sich nämlich, daß sie die von Leverrier entdeckte säkulare Drehung der Merkurbahn im Sinne der Bahnbewegung, welche etwa 45″ im Jahrhundert beträgt qualitativ und quantitativ erklärt, ohne daß irgendwelche besondere
[4] Hypothese zugrunde gelegt werden müßte[2].

Es ergibt sich ferner, daß die Theorie eine stärkere (doppelt so starke) Lichtstrahlenkrümmung durch Gravitationsfelder zur Konsequenz hat als gemäß meinen früheren Untersuchungen.

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[1] In einer bald folgenden Mitteilung wird gezeigt werden, daß jene Hypothese
[3] entbehrlich ist. Wesentlich ist nur, daß eine solche Wahl des Bezugssystems möglich ist, daß die Determinante $|g_{\mu\nu}|$ den Wert -1 annimmt. Die nachfolgende Untersuchung ist hiervon unabhängig.

[2] Über die Unmöglichkeit, die Anomalien der Merkurbewegung auf der Basis der Newtonschen Theorie befriedigend zu erklären, schrieb E. Freundlich jüngst
[5] einen beachtenswerten Aufsatz (Astr. Nachr. 4803, Bd. 201. Juni 1915).

§ 1. Das Gravitationsfeld.

Aus meinen letzten beiden Mitteilungen geht hervor, daß das Gravitationsfeld im Vakuum bei geeignet gewähltem Bezugssystem folgenden Gleichungen zu genügen hat

$$\sum_\alpha \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} + \sum_{\alpha\beta} \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} = 0, \qquad (1)$$

wobei die $\Gamma^\alpha_{\mu\nu}$ durch die Gleichung definiert sind

$$\Gamma^\alpha_{\mu\nu} = -\left\{ \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\} = -\sum_\beta g^{\alpha\beta} \begin{bmatrix} \mu\nu \\ \beta \end{bmatrix} = -\frac{1}{2} \sum_\beta g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right). \qquad (2)$$

Machen wir außerdem die in der letzten Mitteilung begründete Hypothese, daß der Skalar des Energietensors der »Materie« stets verschwinde, so tritt hierzu die Determinantengleichung

$$|g_{\mu\nu}| = -1. \qquad (3)$$

Es befinde sich im Anfangspunkt des Koordinatensystems ein Massenpunkt (die Sonne). Das Gravitationsfeld, welches dieser Massenpunkt erzeugt, kann aus diesen Gleichungen durch sukzessive Approximation berechnet werden.

Es ist indessen wohl zu bedenken, daß die $g_{\mu\nu}$ bei gegebener Sonnenmasse durch die Gleichungen (1) und (3) mathematisch noch nicht vollständig bestimmt sind. Es folgt dies daraus, daß diese Gleichungen bezüglich beliebiger Transformationen mit der Determinante 1 kovariant sind. Es dürfte indessen berechtigt sein, vorauszusetzen, daß alle diese Lösungen durch solche Transformationen aufeinander reduziert werden können, daß sie sich also (bei gegebenen Grenzbedingungen) nur formell, nicht aber physikalisch voneinander unterscheiden. Dieser Überzeugung folgend begnüge ich mich vorerst damit, hier eine Lösung abzuleiten, ohne mich auf die Frage einzulassen, ob es die einzig mögliche sei.

Wir gehen nun in solcher Weise vor. Die $g_{\mu\nu}$ seien in »nullter Näherung« durch folgendes, der ursprünglichen Relativitätstheorie entsprechende Schema gegeben

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\}, \qquad (4)$$

oder kürzere [6]

$$\left.\begin{aligned} g_{\rho\sigma} &= \delta_{\rho\sigma} \\ g_{\rho 4} &= g_{4\rho} = 0 \\ g_{44} &= 1 \end{aligned}\right\}. \qquad (4a)$$

Hierbei bedeuten ρ und σ die Indizes 1, 2, 3; $\delta_{\rho\sigma}$ ist gleich 1 oder 0, je nachdem $\rho = \sigma$ oder $\rho \neq \sigma$ ist.

Wir setzen nun im folgenden voraus, daß sich die $g_{\mu\nu}$ von den in (4a) angegebenen Werten nur um Größen unterscheiden, die klein sind gegenüber der Einheit. Diese Abweichungen behandeln wir als kleine Größen »erster Ordnung«, Funktionen $n$ten Grades dieser Abweichungen als »Größen $n$ter Ordnung«. Die Gleichungen (1) und (3) setzen uns in den Stand, von (4a) ausgehend, durch sukzessive Approximation das Gravitationsfeld bis auf Größen $n$ter Ordnung genau zu berechnen. Wir sprechen in diesem Sinne von der »$n$ten Approximation«; die Gleichungen (4a) bilden die »nullte Approximation«.

Die im folgenden gegebene Lösung hat folgende, das Koordinatensystem festlegende Eigenschaften:

1. Alle Komponenten sind von x_4 unabhängig.
2. Die Lösung ist (räumlich) symmetrisch um den Anfangspunkt des Koordinatensystems, in dem Sinne, daß man wieder auf dieselbe Lösung stößt, wenn man sie einer linearen orthogonalen (räumlichen) Transformation unterwirft.
3. Die Gleichungen $g_{\rho 4} = g_{4\rho} = 0$ gelten exakt (für $\rho = 1$ bis 3).
4. Die $g_{\mu\nu}$ besitzen im Unendlichen die in (4a) gegebenen Werte.

Erste Approximation.

Es ist leicht zu verifizieren, daß in Größen erster Ordnung den Gleichungen (1) und (3) sowie den eben genannten 4 Bedingungen genügt wird durch den Ansatz

$$\left.\begin{aligned} g_{\rho\sigma} &= -\delta_{\rho\sigma} + \alpha\left(\frac{\partial^2 r}{\partial x_\rho \partial x_\sigma} - \frac{\delta_{\rho\sigma}}{r}\right) = -\delta_{\rho\sigma} - \alpha\frac{x_\rho x_\sigma}{r^3} \\ g_{44} &= 1 - \frac{\alpha}{r} \end{aligned}\right\}. \quad (4b)$$

[7] Die $g_{4\rho}$ bzw. $g_{\rho 4}$ sind dabei durch Bedingung 3 festgelegt. r bedeutet die Größe $+\sqrt{x_1^2 + x_2^2 + x_3^2}$, α eine durch die Sonnenmasse bestimmte Konstante.

Daß (3) in Gliedern erster Ordnung erfüllt ist, sieht man sogleich. Um in einfacher Weise einzusehen, daß auch die Feldgleichungen (1) in erster Näherung erfüllt sind, braucht man nur zu beachten, daß bei Vernachlässigung von Größen zweiter und höherer Ordnung die linke Seite der Gleichungen (1) sukzessive durch

$$\sum_\alpha \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha}$$

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix}$$

versetzt werden kann, wobei α nur von 1—3 läuft.

Wie man aus (4b) ersieht, bringt es unsere Theorie mit sich, daß im Falle einer ruhenden Masse die Komponenten g_{11} bis g_{33} bereits in den Größen erster Ordnung von null verschieden sind. Wir werden später sehen, daß hierdurch kein Widerspruch gegenüber Newtons Gesetz (in erster Näherung) entsteht. Wohl aber ergibt sich hieraus [8]
ein etwas anderer Einfluß des Gravitationsfeldes auf einen Lichtstrahl als nach meinen früheren Arbeiten; denn die Lichtgeschwindigkeit ist durch die Gleichung

$$\sum g_{\mu\nu} dx_\mu dx_\nu = 0 \tag{5}$$

bestimmt. Unter Anwendung von Huygens' Prinzip findet man aus (5) und (4b) durch eine einfache Rechnung, daß ein an der Sonne im Abstand Δ vorbeigehender Lichtstrahl eine Winkelablenkung von der Größe $\frac{2\alpha}{\Delta}$ erleidet, während die früheren Rechnungen, bei welchen die Hypothese $\sum T^\mu_\mu = 0$ nicht zugrunde gelegt war, den Wert $\frac{\alpha}{\Delta}$ ergeben hatten. Ein an der Oberfläche der Sonne vorbeigehender [9]
Lichtstrahl soll eine Ablenkung von 1.7″ (statt 0.85″) erleiden. Hingegen bleibt das Resultat betreffend die Verschiebung der Spektrallinien durch das Gravitationspotential, welches durch Herrn Freundlich an den Fixsternen der Größenordnung nach bestätigt wurde, ungeändert bestehen, da dieses nur von g_{44} abhängt. [10]

Nachdem wir die $g_{\mu\nu}$ in erster Näherung erlangt haben, können wir auch die Komponenten $\Gamma^\sigma_{\mu\nu}$ des Gravitationsfeldes in erster Näherung [11]
berechnen. Aus (2) und (4b) ergibt sich

$$\Gamma^\tau_{\rho\sigma} = -\alpha\left(\delta_{\rho\sigma}\frac{x_\tau}{r^3} - \frac{3}{2}\frac{x_\rho x_\sigma x_\tau}{r^5}\right), \tag{6a}$$ [12]

wobei ρ, σ, τ irgendwelche der Indizes 1, 2, 3 bedeuten,

$$\Gamma^\sigma_{44} = \Gamma^4_{4\sigma} = -\frac{\alpha}{2}\frac{x_\sigma}{r^3}, \tag{6b}$$

wobei σ den Index 1, 2 oder 3 bedeutet. Diejenigen Komponenten, in welchen der Index 4 einmal oder dreimal auftritt, verschwinden.

Zweite Approximation.

Es wird sich nachher ergeben, daß wir nur die drei Komponenten Γ^σ_{44} in Größen zweiter Ordnung genau zu ermitteln brauchen, um die Planetenbahnen mit dem entsprechenden Genauigkeitsgrade ermitteln zu können. Hierfür genügt uns die letzte Feldgleichung zusammen mit den allgemeinen Bedingungen, welche wir unserer Lösung auferlegt haben. Die letzte Feldgleichung

$$\sum_{\sigma} \frac{\partial \Gamma_{44}^{\sigma}}{\partial x_{\sigma}} + \sum_{\sigma\tau} \Gamma_{4\tau}^{\sigma} \Gamma_{4\sigma}^{\tau} = 0$$

geht mit Rücksicht auf (6b) bei Vernachlässigung von Größen dritter und höherer Ordnung über in

[13]
$$\sum_{\sigma} \frac{\Gamma_{44}^{\sigma}}{\partial x_{\sigma}} = \frac{\alpha^2}{2r^4}.$$

Hieraus folgern wir mit Rücksicht auf (6b) und die Symmetrieeigenschaften unserer Lösung

$$\Gamma_{44}^{\sigma} = -\frac{\alpha}{2} \frac{x_{\sigma}}{r^3} \left(1 - \frac{\alpha}{r}\right). \tag{6c}$$

§ 2. Die Planetenbewegung.

Die von der allgemeinen Relativitätstheorie gelieferten Bewegungsgleichungen des materiellen Punktes im Schwerefelde lauten

$$\frac{d^2 x_{\nu}}{ds^2} = \sum_{\sigma\tau} \Gamma_{\sigma\tau}^{\nu} \frac{dx_{\sigma}}{ds} \frac{dx_{\tau}}{ds}. \tag{7}$$

Aus diesen Gleichungen folgern wir zunächst, daß sie die NEWTONschen Bewegungsgleichungen als erste Näherung enthalten. Wenn nämlich die Bewegung des Punktes mit gegen die Lichtgeschwindigkeit kleiner Geschwindigkeit stattfindet, so sind dx_1, dx_2, dx_3 klein gegen dx_4. Folglich bekommen wir eine erste Näherung, indem wir auf der rechten Seite jeweilen nur das Glied $\sigma = \tau = 4$ berücksichtigen. Man erhält dann mit Rücksicht auf (6b)

$$\left.\begin{aligned} \frac{d^2 x_{\nu}}{ds^2} &= \Gamma_{44}^{\nu} = -\frac{\alpha}{2} \frac{x_{\nu}}{r^3} \quad (\nu = 1, 2, 3) \\ \frac{d^2 x_4}{ds^2} &= 0 \end{aligned}\right\}. \tag{7a}$$

Diese Gleichungen zeigen, daß man für eine erste Näherung $s = x_4$ setzen kann. Dann sind die ersten drei Gleichungen genau die NEWTONschen. Führt man in der Bahnebene Polargleichungen r, ϕ ein, so liefern der Energie- und der Flächensatz bekanntlich die Gleichungen

$$\left.\begin{aligned} \frac{1}{2} u^2 + \Phi &= A \\ r^2 \frac{d\phi}{ds} &= B \end{aligned}\right\}, \tag{8}$$

wobei A und B die Konstanten des Energie- bzw. Flächensatzes bedeuten, wobei zur Abkürzung

$$\left.\begin{aligned} \Phi &= -\frac{\alpha}{2r} \\ u^2 &= \frac{dr^2 + r^2 d\phi^2}{ds^2} \end{aligned}\right\} \qquad (8a)$$

gesetzt ist.

Wir haben nun die Gleichungen (7) um eine Größenordnung genauer auszuwerten. Die letzte der Gleichungen (7) liefert dann zusammen mit (6b)

$$\frac{d^2 x_4}{ds^2} = 2\sum_{\sigma} \Gamma^4_{\sigma 4} \frac{dx_\sigma}{ds}\frac{dx_4}{ds} = -\frac{dg_{44}}{ds}\frac{dx_4}{ds}$$

oder in Größen erster Ordnung genau

$$\frac{dx_4}{ds} = 1 + \frac{\alpha}{r}. \qquad (9)$$

Wir wenden uns nun zu den ersten drei Gleichungen (7). Die rechte Seite liefert

a) für die Indexkombination $\sigma = \tau = 4$

$$\Gamma^\nu_{44}\left(\frac{dx_4}{ds}\right)^2$$

oder mit Rücksicht auf (6c) und (9) in Größen zweiter Ordnung genau

$$-\frac{\alpha}{2}\frac{x_\nu}{r^3}\left(1 + \frac{\alpha}{r}\right),$$

b) für die Indexkombinationen $\sigma \neq 4$ $\tau \neq 4$ (welche allein noch in Betracht kommen) mit Rücksicht darauf, daß die Produkte $\frac{dx_\sigma}{ds}\frac{dx_\tau}{ds}$ [14]
mit Rücksicht auf (8) als Größen erster Ordnung anzusehen sind[1], ebenfalls auf Größen zweiter Ordnung genau

$$-\frac{\alpha x_\nu}{r^3}\sum_{\sigma\tau}\left(\delta_{\sigma\tau} - \frac{3}{2}\frac{x_\sigma x_\tau}{r^2}\right)\frac{dx_\sigma}{ds}\frac{dx_\tau}{ds}.$$

Die Summation ergibt

$$-\frac{\alpha x_\nu}{r^3}\left(u^2 - \frac{3}{2}\left(\frac{dr}{ds}\right)^2\right).$$

---

[1] Diesem Umstand entsprechend können wir uns bei den Feldkomponenten $\Gamma^\nu_{\sigma\tau}$ mit der in Gleichung (6a) gegebenen ersten Näherung begnügen.

Mit Rücksicht hierauf erhält man für die Bewegungsgleichungen die in Größen zweiter Ordnung genaue Form

$$\frac{d^2x_\nu}{ds^2} = -\frac{\alpha}{2}\frac{x_\nu}{r^3}\left(1 + \frac{\alpha}{r} + 2u^2 - 3\left(\frac{dr}{ds}\right)^2\right), \qquad (7\,\text{b})$$

welche zusammen mit (9) die Bewegung des Massenpunktes bestimmt. Nebenbei sei bemerkt, daß (7b) und (9) für den Fall der Kreisbewegung keine Abweichungen vom dritten KEPLERschen Gesetze ergeben.

Aus (7b) folgt zunächst die exakte Gültigkeit der Gleichung

$$r^2\frac{d\phi}{ds} = B, \qquad (10)$$

wobei B eine Konstante bedeutet. Der Flächensatz gilt also in Größen zweiter Ordnung genau, wenn man die »Eigenzeit« des Planeten zur Zeitmessung verwendet. Um nun die säkulare Drehung der Bahnellipse aus (7b) zu ermitteln, ersetzt man die Glieder erster Ordnung in der Klammer der sechsten Seite am vorteilhaftesten vermittels (10) und der ersten der Gleichungen (8), durch welches Vorgehen die Glieder zweiter Ordnung auf der rechten Seite nicht geändert werden. Die Klammer nimmt dadurch die Form an

$$\left(1 - 2A + \frac{3B^2}{r^2}\right).$$

Wählt man endlich $s\sqrt{1-2A}$ als Zeitvariable, und nennt man letztere wieder s, so hat man bei etwas geänderter Bedeutung der Konstanten B:

[15]
$$\left.\begin{aligned}\frac{d^2x_\nu}{ds^2} &= -\frac{\partial\Phi}{\partial x_\nu}\\ \Phi &= -\frac{\alpha}{2}\left[1 + \frac{B^2}{r^2}\right]\end{aligned}\right\}. \qquad (7\,\text{c})$$

Bei der Bestimmung der Bahnform geht man nun genau vor wie im NEWTONschen Falle. Aus (7c) erhält man zunächst

$$\frac{dr^2 + r^2d\phi^2}{ds^2} = 2A - 2\Phi.$$

Eliminiert man aus dieser Gleichung ds mit Hilfe von (10), so ergibt sich, indem man mit x die Größe $\frac{1}{r}$ bezeichnet:

$$\left(\frac{dx}{d\phi}\right)^2 = \frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3, \qquad (11)$$

welche Gleichung sich von der entsprechenden der NEWTONschen Theorie nur durch das letzte Glied der rechten Seite unterscheidet.

Der vom Radiusvektor zwischen dem Perihel und dem Aphel beschriebene Winkel wird demnach durch das elliptische Integral

$$\phi = \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3}},$$

wobei α_1 und α_2 diejenigen Wurzeln der Gleichung

$$\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3 = 0$$

bedeuten, welchen sehr benachbarte Wurzeln derjenigen Gleichung entsprechen, die aus dieser durch Weglassen des letzten Gliedes entsteht.

Hierfür kann mit der von uns zu fordernden Genauigkeit gesetzt werden

$$\phi = [1 + \alpha(\alpha_1 + \alpha_2)] \cdot \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)(1-\alpha x)}}$$ [16]

oder nach Entwicklung von $(1-\alpha x)^{-\frac{1}{2}}$

$$\phi = [1 + \alpha(\alpha_1 + \alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{\left(1 + \frac{\alpha}{2}x\right)dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}}.$$

Die Integration liefert

$$\phi = \pi\left[1 + \frac{3}{4}\alpha(\alpha_1 + \alpha_2)\right],$$

oder, wenn man bedenkt, daß α_1 und α_2 die reziproken Werte der maximalen bzw. minimalen Sonnendistanz bedeuten,

$$\phi = \pi\left(1 + \frac{3}{2}\frac{\alpha}{a(1-e^2)}\right). \qquad (12)$$

Bei einem ganzen Umlauf rückt also das Perihel um

$$\varepsilon = 3\pi \frac{\alpha}{a(1-e^2)} \qquad (13)$$

im Sinne der Bahnbewegung vor, wenn mit a die große Halbachse, mit e die Exzentrizität bezeichnet wird. Führt man die Umlaufszeit T

(in Sekunden) ein, so erhält man, wenn c die Lichtgeschwindigkeit in cm/sec. bedeutet:

$$\varepsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1-e^2)}. \qquad (14)$$

Die Rechnung liefert für den Planeten Merkur ein Vorschreiten des Perihels um 43″ in hundert Jahren, während die Astronomen 45″ ± 5″ als unerklärten Rest zwischen Beobachtungen und NEWTONscher Theorie
[17] angeben. Dies bedeutet volle Übereinstimmung.

Für Erde und Mars geben die Astronomen eine Vorwärtsbewegung von 11″ bzw. 9″ in hundert Jahren an, während unsere Formel nur 4″ bzw. 1″ liefert. Es scheint jedoch diesen Angaben wegen der zu geringen Exzentrizität der Bahnen jener Planeten ein geringer Wert eigen zu sein. Maßgebend für die Sicherheit der Konstatierung der Perihelbewegung ist ihr Produkt mit der Exzentrizität $\left(e\frac{d\pi}{dt}\right)$. Be-
[18] trachtet man die für diese Größe von NEWCOMB angegebenen Werte

	$e\frac{d\pi}{dt}$
Merkur	8.48″ ± 0.43
Venus	−0.05 ± 0.25
Erde	0.10 ± 0.13
Mars	0.75 ± 0.35,

welche ich Hrn. Dr. FREUNDLICH verdanke, so gewinnt man den Eindruck, daß ein Vorrücken des Perihels überhaupt nur für Merkur wirklich nachgewiesen ist. Ich will jedoch ein endgültiges Urteil hierüber gerne den Fachastronomen überlassen.

Notes

The original text in German of the translated Einstein's paper was included without translation into Volume 6 of *The Collected Papers of Albert Einstein* (further *ColPap* for short); Volume **6** (1996), *Doc.*24. *A. J. Knox, M. J. Klein, and R. Schulmann, Editors. Princeton University Press.* These papers are assigned numbers *Doc. namber*. The paper under discussion is referred to as *Doc.*24. It was provided with Editor's Notes numbered on the paper margins. We retain these numbers in the attached original paper (*Doc.*24) for making our *Notes* and comments on the selected spots in addition to our other comments, sometimes, with a reference to our work *"General Relativity Problem of Mercury's Perihelion Advance Revisited", arXiv, physics. gen-ph, 1008.1811v1, Aug. 2010* (further [Van10] for short).

As a matter of fact, we found another edition of *ColPap* in which Vol. 6 (1997) contains the above Einstein's paper, *Doc.*24, translated into English by Brian Doyle, and reprinted from *A Source Book in Astronomy and Astrophysics, 1900 - 1975, edited by Kenneth R. Lang and Owen Gingerich.* There are Editors' Notes numbers on pages but actual Notes are not provided, and mistakes made in the original publication are not corrected in the translation.

In the following *Notes*, corrected mistakes are listed.

1, 2. Einstein refers to the paper, *Doc.*21 in Vol. 6, *"On the General Theory of Relativity", 1915.* There he makes a further reference to the similar paper (1914) in the *Sitzungsberichte*, *Doc.*9. He is concerned about a return to the General Covariance Principle after partly abandoning it in works with Grossman (*the Entwurf theory*). Here he states that the hypothesis about the vanishing trace of "matter" tensor having the metric determinant being equal to unity is in accord with the General Covariance of field equations. Next Einstein refers to the Addendum which is titled *"On the General Theory of Relativity, Addendum", 1915*, *Doc.*22. There he says: "In the following we assume the conditions $\sum_\nu^\nu T_\nu^\nu = 0$ really to be generally true" and "Then, however, we are also entitled to add to our previous field equation the limiting condition $\sqrt{-g} = 1$. See also the paper *"The Field Equations of Gravitation", November 25, 1915*, *Doc.*25.

3. Here (in Einstein's footnote) is the statement about the change of the

above theoretical premises.

4, 5. See the historical note by Editors of *ColPap*, also [$Van10$],

6. In the original, a minus sign on the r.h.s. of the first equation is missing (corrected).

7. In the original, ν should be r (corrected).

8. See the note by Editors of *ColPap* about historical discussion of the issue in *Norton, 1984*.

9, 10. See the note by Editors of *CalPap* concerning the calculations, reference to *Doc.*30, 1916, *The Foundations of the General Theory of Relativity*, and the condition $\sum_{\nu}^{\nu} T_{\nu}^{\nu} = 0$, *Doc.*25, retracted. Also about history of red-shift observations.

12. In the original, r^2 in the first term on the r.h.s. should be r^3 (corrected).

13. In the original, a minus sign is missing on the r.h.s. of the equation (corrected).

14. In the original, x_r should be x_τ (corrected).

15. In the second equation, r is missing: the factor 2 should be $2r$ (corrected).

16. In the first two equations for ϕ, the factor α in front of the integral should be $\alpha/2$ (corrected).

17. See the historical note by Editors of *ColPap* about the data source *Newcomb 1895*.

B Schwarzschild's letter to Einstein

Letter from K Schwarzschild to A Einstein dated 22 December 1915

(*ColPap*, vol. 8a, *Doc.*169, posted on Internet)

22.XII.15.

Verehrter Herr Einstein!

Um mit Ihrer Gravitationstheorie vertraut zu werden, habe ich mich näher mit dem von Ihnen in der Arbeit über das Merkurperihel gestellte und in 1. Näherung gelöste Problem beschäftigt. Zunächst machte mich ein Umstand sehr konfus. Ich fand für die erste Näherung der Koeffizienten $g_{\mu\nu}$ außer ihrer Lösung noch folgende zweite:

$$g_{\rho\sigma} = -\frac{\beta x_\rho x_\sigma}{r^5} + \delta_{\rho\sigma}\left[\frac{\beta}{3r^3}\right] \qquad g_{44} = 1$$

Danach hätte es außer Ihrem α noch eine zweite gegeben und das Problem wäre physikalisch unbestimmt. Daraufhin machte ich einmal auf gut Glück den Versuch einer vollständigen Lösung. Eine nicht zu große Rechnerei ergab folgendes Resultat: Es gibt nur ein Linienelement, das Ihre Bedingungen 1) bis 4) nebst Feld- und Determinantengl. erfüllt und im Nullpunkt und nur im Nullpunkt singulär ist.

Sei:

$$x_1 = r\cos\phi\cos\theta \qquad x_2 = r\sin\phi\cos\theta \qquad x_3 = r\sin\theta$$

$$R = (r^3+\alpha^3)^{1/3} = r\left(1 + \frac{1}{3}\frac{\alpha^3}{r^3} + \dots\right)$$

dann lautet das Linienelement:

$$ds^2 = \left(1 - \frac{\gamma}{R}\right)dt^2 - \frac{dR^2}{1-\frac{\gamma}{R}} - R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

R, θ, ϕ sind keine „erlaubten“ Koordinaten, mit denen man die Feldgleichungen bilden dürfte, weil sie nicht die Determinante 1 haben, aber das Linienelement schreibt sich in ihnen am schönsten.

Die Gleichung der Bahnkurve bleibt genau die von Ihnen in erster Näherung erhaltene (11), nur muß man unter x nicht $\frac{1}{r}$, sondern $\frac{1}{R}$ verstehen, was ein Unterschied von der Ordnung 10^{-12} ist, also praktisch absolut gleichgültig.

Die Schwerigkeit mit den zwei willkürlichen Konstanten α und β, welche die erste Näherung gab, löst sich dahin, daß β einen bestimmten Wert von der Ordnung α^4 haben muß, so wie α gegeben ist, sonst würde die Lösung bei Fortsetzung der Näherungen divergent.

Es ist also auch die Eindeutigkeit Ihres Problems in schönster Ordnung.

Es ist eine ganz wunderbare Sache, daß von einer so abstrakten Idee aus die Erklärung der Merkuranomalie so zwingend herauskommt.

Wie Sie sehen, meint es der Krieg freundlich mit mir, indem er mir trotz heftigen Geschützfeuers in der durchaus terrestrischer Entfernung diesen Spaziergang in dem von Ihrem Ideenlande erlaubte.